

THE PROOF OF THE l^2 DECOUPLING CONJECTURE

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ABSTRACT. We prove the l^2 Decoupling Conjecture for compact hypersurfaces with positive definite second fundamental form. This has a wide range of important consequences. One of them is the validity of the Discrete Restriction Conjecture, which (up to N^ϵ losses) implies the full range of expected $L^p_{x,t}$ Strichartz estimates for both classical and irrational tori. Another one is an improvement in the range for the discrete restriction theory for lattice points on the sphere. Various applications in Additive Combinatorics, Incidence Geometry and Number Theory are also discussed. Our argument relies on the interplay between linear and multilinear theory.

1. THE l^2 DECOUPLING THEOREM

Let S be a compact C^2 hypersurface in \mathbb{R}^n with positive definite second fundamental form. Examples include the sphere S^{n-1} and the truncated (elliptic) paraboloid

$$P^{n-1} := \{(\xi_1, \dots, \xi_{n-1}, \xi_1^2 + \dots + \xi_{n-1}^2) \in \mathbb{R}^n : |\xi_i| \leq 1/2\}.$$

Unless specified otherwise, we will implicitly assume throughout the whole paper that $n \geq 2$. We will write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The implicit constants hidden inside the symbols \lesssim and \sim will in general depend on fixed parameters such as p , n , α and sometimes on variable parameters such as ϵ . We will not record the dependence on the fixed parameters.

Let \mathcal{N}_δ be the δ neighborhood of P^{n-1} and let \mathcal{P}_δ be a finitely overlapping cover of \mathcal{N}_δ with curved regions θ of the form

$$\theta = \{(\xi_1, \dots, \xi_{n-1}, \eta + \xi_1^2 + \dots + \xi_{n-1}^2) : (\xi_1, \dots, \xi_{n-1}) \in C_\theta, |\eta| \leq 2\delta\}, \quad (1)$$

where C_θ runs over all cubes $c + [-\frac{\delta^{1/2}}{2}, \frac{\delta^{1/2}}{2}]^{n-1}$ with $c \in \frac{\delta^{1/2}}{2}\mathbb{Z}^{n-1} \cap [-1/2, 1/2]^{n-1}$. Note that each θ sits inside a $\sim \delta^{1/2} \times \dots \times \delta^{1/2} \times \delta$ rectangular box. It is also important to realize that the normals to these boxes are $\sim \delta^{1/2}$ separated. A similar decomposition exists for any S as above and we will use the same notation \mathcal{P}_δ for it. We will denote by f_θ the Fourier restriction of f to θ .

Our main result is the proof of the following l^2 Decoupling Theorem.

Theorem 1.1. *Let S be a compact C^2 hypersurface in \mathbb{R}^n with positive definite second fundamental form. If $\text{supp}(\hat{f}) \subset \mathcal{N}_\delta$ then for $p \geq \frac{2(n+1)}{n-1}$ and $\epsilon > 0$*

$$\|f\|_p \lesssim_\epsilon \delta^{-\frac{n-1}{4} + \frac{n+1}{2p} - \epsilon} \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^2 \right)^{1/2}. \quad (2)$$

Key words and phrases. discrete restriction estimates, Strichartz estimates, additive energy.

The first author is partially supported by the NSF grant DMS-1301619. The second author is partially supported by the NSF Grant DMS-1161752.

Theorem 1.1 has been proved in [19] for $p > 2 + \frac{8}{n-1} - \frac{4}{n(n-1)}$. A standard construction is presented in [19] to show that, up to the $\delta^{-\epsilon}$ term, the exponent of δ is optimal. We point out that Wolff [31] has initiated the study of l^p decouplings, $p > 2$ in the case of the cone. His work provides part of the inspiration for our paper.

A localization argument and interpolation between $p = \frac{2(n+1)}{n-1}$ and the trivial bound for $p = 2$ proves the subcritical estimate

$$\|f\|_p \lesssim_{\epsilon} \delta^{-\epsilon} \left(\sum_{\theta \in \mathcal{P}_{\delta}} \|f_{\theta}\|_p^2 \right)^{1/2}, \quad (3)$$

when $2 \leq p < \frac{2(n+1)}{n-1}$. Estimate (3) is false for $p < 2$. This can easily be seen by testing it with functions of the form $f_{\theta}(x) = g_{\theta}(x + c_{\theta})$, where $\text{supp}(\widehat{g}_{\theta}) \subset \theta$ and the numbers c_{θ} are very far apart from each other.

Inequality (3) has been recently proved by the first author for $p = \frac{2n}{n-1}$ in [10], using a variant of the induction on scales from [13] and the multilinear restriction Theorem 6.1.

An argument similar to the one in [19] was used in [17] to prove Theorem 1.1 for $p > \frac{2(n+2)}{n-1}$, by interpolating Wolff's machinery with the estimate $p = \frac{2n}{n-1}$ from [10]. This range is better than the one in [19] due to the use of multilinear theory as opposed to bilinear theory.¹

We mention briefly that there is a stronger form of decoupling, sometimes referred to as *square function estimate*, which predicts that

$$\|f\|_p \lesssim_{\epsilon} \delta^{-\epsilon} \left\| \left(\sum_{\theta \in \mathcal{P}_{\delta}} |f_{\theta}|^2 \right)^{1/2} \right\|_p, \quad (4)$$

in the slightly smaller range $2 \leq p \leq \frac{2n}{n-1}$. When $n = 2$ this easily follows via a geometric argument. Minkowski's inequality shows that (4) is indeed stronger than (3) in the range $2 \leq p \leq \frac{2n}{n-1}$. This is also confirmed by the lack of any results for (4) when $n \geq 3$. Our methods do not seem to enable any progress on (4).

It is reasonable to hope that in the subcritical regime (3) one may be able to replace $\delta^{-\epsilon}$ by a constant $C_{p,n}$ independent of δ . This is indeed known when $n = 2$ and $p \leq 4$, but seems to be in general an extremely difficult question. To the authors' knowledge, no other examples of $2 < p < \frac{2(n+1)}{n-1}$ are known for when this holds.

In Section 5 we introduce a multilinear version of the decoupling inequality (2) and show that the multilinear and the linear theories are essentially equivalent. This in itself is not enough to prove Theorem 1.1, as Theorem 6.1 gives multilinear decoupling only in the range $2 \leq p \leq \frac{2n}{n-1}$. To bridge the gap between $\frac{2n}{n-1}$ and $\frac{2(n+1)}{n-1}$, in Section 6 we refine our analysis based on the multilinear theory. In particular we set up an induction on scales argument that makes use of Theorem 6.1 at each step of the iteration, rather than once.

Theorem 1.1 immediately implies the validity of the Discrete Restriction Conjecture in the expected range, see Theorem 2.2 below. This in turn has a wide range of interesting consequences that are detailed in Section 2. First, we get the full range of expected $L_{x,t}^p$ Strichartz estimates for both classical and irrational tori. Second, we derive sharp estimates on the additive energies of various sets. These can be rephrased as incidence

¹While both the bilinear theorem in [29] and the multilinear theorem in [2] are sharp, the latter one is "morally" stronger

geometry problems and in some cases we are not aware about an alternative approach. While our theorems successfully address the case of "nicely separated" points, some intriguing questions are left open for arbitrary points.

A third type of applications includes sharp (up to N^ϵ losses) estimates for the number of solutions of various diophantine inequalities. This is rather surprising given the fact that our methods do not rely on any number theory. We believe that they provide a new angle by means of our use of induction on scales and the topology of \mathbb{R}^n . Indeed, the Multilinear Restriction Theorem 6.1 that we use repeatedly in the proof of our main Theorem 1.1 relies at its core on the multilinear Kakeya phenomenon, which is of topological nature (see [22], [15]).

Finally, we use Theorem 2.2 to improve the range from [11], [12] in the discrete restriction problem for lattice points on the sphere.

We point out in Subsection 2.1 that there is no analogous l^2 decoupling theory if the second fundamental form of S is not semidefinite. One can however develop a robust l^p decoupling theory in the case of nonzero Gaussian curvature that has some striking number theoretical applications. Also, the l^2 decoupling theory can be extended to the case of the cone. These further applications will be discussed elsewhere.

Acknowledgment. The authors have benefited from helpful conversations with Nets Katz and Andreas Seeger. The second author would like to thank his student Fangye Shi for a careful reading of the manuscript and for pointing out a few typos.

2. FIRST APPLICATIONS

In this section we present the first round of applications of our decoupling theory. Additional applications will appear elsewhere.

2.1. The discrete restriction phenomenon. To provide some motivation we recall the Stein-Tomas Restriction Theorem.

Theorem 2.1. *Let S be a compact C^2 hypersurface in \mathbb{R}^n with nonzero Gaussian curvature and let $d\sigma$ denote the natural surface measure on S . Then for $p \geq \frac{2(n+1)}{n-1}$ and $f \in L^2(S, d\sigma)$ we have*

$$\|\widehat{fd\sigma}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^2(S)}.$$

Note that this result only needs nonzero Gaussian curvature. We will use the notation $e(a) = e^{2\pi i a}$. For fixed $p \geq \frac{2(n+1)}{n-1}$, it is an easy exercise to see that this theorem is equivalent with the statement that

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x) \right|^p \right)^{1/p} \lesssim \delta^{\frac{n}{2p} - \frac{n-1}{4}} \|a_\xi\|_{l^2(\Lambda)},$$

for each $0 \leq \delta \leq 1$, each $a_\xi \in \mathbb{C}$, each ball $B_R \subset \mathbb{R}^n$ of radius $R \sim \delta^{-1/2}$ and each $\delta^{1/2}$ separated set $\Lambda \subset S$. Thus, the Stein-Tomas Theorem measures the average L^p

oscillations of exponential sums at spatial scale equal to the inverse of the separation of the frequencies. It will be good to keep in mind that for each $R \gtrsim \delta^{-1/2}$

$$\left\| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x) \right\|_{L^2(B_R)} \sim |B_R|^{1/2} \|a_\xi\|_2, \quad (5)$$

as can be seen using Plancherel's Theorem.

The discrete restriction phenomenon consists in the existence of stronger cancellations at the larger scale $R \gtrsim \delta^{-1}$. We prove the following.

Theorem 2.2. *Let S be a compact C^2 hypersurface in \mathbb{R}^n with positive definite second fundamental form. Let $\Lambda \subset S$ be a $\delta^{1/2}$ -separated set and let $R \gtrsim \delta^{-1}$. Then for each $\epsilon > 0$*

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x) \right|^p \right)^{1/p} \lesssim_\epsilon \delta^{\frac{n+1}{2p} - \frac{n-1}{4} - \epsilon} \|a_\xi\|_2 \quad (6)$$

if $p \geq \frac{2(n+1)}{n-1}$.

It has been observed in [10] that Theorem 1.1 for a given p implies (6) for the same p . Here is a sketch of the argument. First, note that the statement

$$\|f\|_p \lesssim \delta^{c_p} \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^2 \right)^{1/2}, \text{ whenever } \text{supp}(\hat{f}) \subset \mathcal{N}_\delta$$

easily implies that for each $g : S \rightarrow \mathbb{C}$ and $R \gtrsim \delta^{-1}$

$$\left(\int_{B_R} |\widehat{gd\sigma}|^p \right)^{1/p} \lesssim \delta^{c_p} \left(\sum_{\theta \in \mathcal{P}_\delta} \|\widehat{g_\theta d\sigma}\|_{L^p(w_{B_R})}^2 \right)^{1/2}, \quad (7)$$

where here $g_\theta = g|_\theta$ is the restriction of g to the $\delta^{1/2}$ -cap θ on S . See Remark 5.2. Also, throughout the paper we write

$$\|f\|_{L^p(w_{B_R})} = \left(\int_{\mathbb{R}^n} |f(x)|^p w_{B_R}(x) dx \right)^{1/p}$$

for weights w_{B_R} which are Fourier supported in $B(0, \frac{1}{R})$ and satisfy

$$1_{B_R}(x) \lesssim w_{B_R}(x) \leq \left(1 + \frac{|x - c(B_R)|}{R} \right)^{-10n}. \quad (8)$$

It now suffices to use $g = \sum_{\xi \in \Lambda} a_\xi \sigma(U(\xi, \tau))^{-1} 1_{U(\xi, \tau)}$ in (7), where $U(\xi, \tau)$ is a τ -cap on S centered at ξ , and to let $\tau \rightarrow 0$.

Using (6) with $p = \frac{2(n+1)}{n-1}$ and Hölder's inequality we determine that

$$\delta^\epsilon \|a_\xi\|_2 \lesssim_\epsilon \left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x) \right|^p \right)^{1/p} \lesssim_\epsilon \delta^{-\epsilon} \|a_\xi\|_2 \quad (9)$$

for $1 \leq p \leq \frac{2(n+1)}{n-1}$ and $R \gtrsim \delta^{-1}$. We mention that prior to our current work, the only known results for (6) and (9) were the ones in the range where Theorem 1.1 was known.

Theorem 2.2 indicates that there is no interesting l^2 decoupling for the hyperbolic paraboloid

$$\{(\xi_1, \xi_2, \xi_1^2 - \xi_2^2) \in \mathbb{R}^3 : |\xi_i| \leq 1/2\}.$$

Indeed, note that there is no reverse Hölder's inequality for exponential sums if the frequencies are equidistant points on a line. Thus the decoupling regions² on S would have to be chosen in such a way that each line on S intersects $O(1)$ regions. The issue then is the fact that this surface is doubly ruled.

2.2. Strichartz estimates for the classical and irrational tori. The discrete restriction phenomenon has mostly been investigated in the special case when the frequency points Λ come from a lattice. There is extra motivation in considering this case coming from PDEs, where there is interest in establishing Strichartz estimates for the Schrödinger equation on the torus. Prior to the current work, the best known result for the paraboloid

$$P^{n-1}(N) := \{\xi := (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n : \xi_n = \xi_1^2 + \dots + \xi_{n-1}^2, |\xi_1|, \dots, |\xi_{n-1}| \leq N\}$$

was obtained by the first author [9], [10]. We recall this result below.

Theorem 2.3 (Discrete restriction: the lattice case (paraboloid)). *Let $a_\xi \in \mathbb{C}$ and $\epsilon > 0$. Then*

(i) *if $n \geq 4$ we have*

$$\left\| \sum_{\xi \in P^{n-1}(N)} a_\xi e(\xi \cdot x) \right\|_{L^p(\mathbb{T}^n)} \lesssim_\epsilon N^{\frac{n-1}{2} - \frac{n+1}{p} + \epsilon} \|a_\xi\|_{l^2},$$

for $p \geq \frac{2(n+2)}{n-1}$ and

$$\left\| \sum_{\xi \in P^{n-1}(N)} a_\xi e(\xi \cdot x) \right\|_{L^p(\mathbb{T}^n)} \lesssim_\epsilon N^\epsilon \|a_\xi\|_{l^2},$$

for $1 \leq p \leq \frac{2n}{n-1}$.

(ii) *If $n = 2, 3$ then*

$$\left\| \sum_{\xi \in P^{n-1}(N)} a_\xi e(\xi \cdot x) \right\|_{L^p(\mathbb{T}^n)} \lesssim_\epsilon N^\epsilon \|a_\xi\|_{l^2},$$

for $p = \frac{2(n+1)}{n-1}$.

The proof of (i) combines the implementation of the Stein-Tomas argument via the circle method with the inequality (3) proved in [10]. The argument for (ii) is much easier, it uses the fact that circles in the plane contain "few" lattice points. It has been conjectured in [9] that (ii) should hold for $n \geq 4$, too. This is easily seen to be sharp, up to the N^ϵ term. We will argue below that our Theorem 2.2 implies this conjecture, in fact a more general version of it.

The analogous question for the more general irrational tori has been recently investigated in [8], [16], [18] and [21]. More precisely, fix $\frac{1}{2} < \theta_1, \dots, \theta_{n-1} < 2$. For $\phi \in L^2(\mathbb{T}^{n-1})$ consider its Laplacian

$$\begin{aligned} \Delta \phi(x_1, \dots, x_{n-1}) = \\ \sum_{(\xi_1, \dots, \xi_{n-1}) \in \mathbb{Z}^{n-1}} (\xi_1^2 \theta_1 + \dots + \xi_{n-1}^2 \theta_{n-1}) \hat{\phi}(\xi_1, \dots, \xi_{n-1}) e(\xi_1 x_1 + \dots + \xi_{n-1} x_{n-1}) \end{aligned}$$

²such as caps on the sphere or strips along light rays on the cone

on the irrational torus $\prod_{i=1}^{n-1} \mathbb{R}/(\theta_i \mathbb{Z})$. Let also

$$e^{it\Delta} \phi(x_1, \dots, x_{n-1}, t) = \sum_{(\xi_1, \dots, \xi_{n-1}) \in \mathbb{Z}^{n-1}} \hat{\phi}(\xi_1, \dots, \xi_{n-1}) e(x_1 \xi_1 + \dots + x_{n-1} \xi_{n-1} + t(\xi_1^2 \theta_1 + \dots + \xi_{n-1}^2 \theta_{n-1})).$$

We prove

Theorem 2.4 (Strichartz estimates for irrational tori). *Let $\phi \in L^2(\mathbb{T}^{n-1})$ with $\text{supp}(\hat{\phi}) \subset [-N, N]^{n-1}$. Then for each $\epsilon > 0$, $p \geq \frac{2(n+1)}{n-1}$ and each interval $I \subset \mathbb{R}$ with $|I| \gtrsim 1$ we have*

$$\|e^{it\Delta} \phi\|_{L^p(\mathbb{T}^{n-1} \times I)} \lesssim_\epsilon N^{\frac{n-1}{2} - \frac{n+1}{p} + \epsilon} |I|^{1/p} \|\phi\|_2, \quad (10)$$

and the implicit constant does not depend on I , N and θ_i .

Proof For $-N \leq \xi_1, \dots, \xi_{n-1} \leq N$ define $\eta_i = \frac{\theta_i^{1/2} \xi_i}{4N}$ and $a_\eta = \hat{\phi}(\xi)$. A simple change of variables shows that

$$\int_{\mathbb{T}^{n-1} \times I} |e^{it\Delta} \phi|^p \lesssim \frac{1}{N^{n+1}} \int_{\substack{|y_1|, \dots, |y_{n-1}| \leq 8N \\ y_n \in I_{N^2}}} \left| \sum_{\eta_1, \dots, \eta_{n-1}} a_\eta e(y_1 \eta_1 + \dots + y_{n-1} \eta_{n-1} + y_n(\eta_1^2 + \dots + \eta_{n-1}^2)) \right|^p dy_1 \dots dy_n,$$

where I_{N^2} is an interval of length $\sim N^2|I|$. By periodicity in the y_1, \dots, y_{n-1} variables we bound the above by

$$\frac{1}{N^{n+1}(N|I|)^{n-1}} \int_{B_{N^2|I|}} \left| \sum_{\eta_1, \dots, \eta_{n-1}} a_\eta e(y_1 \eta_1 + \dots + y_{n-1} \eta_{n-1} + y_n(\eta_1^2 + \dots + \eta_{n-1}^2)) \right|^p dy_1 \dots dy_n,$$

for some ball $B_{N^2|I|}$ of radius $\sim N^2|I|$. Our result will follow once we note that the points

$$(\eta_1, \dots, \eta_{n-1}, \eta_1^2 + \dots + \eta_{n-1}^2)$$

are $\sim \frac{1}{N}$ separated on P^{n-1} and then apply Theorem 2.2 with $R \sim N^2|I|$. ■

The case $\theta_1 = \dots = \theta_{n-1} = 1$ corresponds to the classical torus. Note that without additional assumptions on θ_i , Theorem 2.4 is sharp up to the ϵ loss, as the lattice case $\theta_1 = \dots = \theta_{n-1} = 1$ shows. It may come as a surprise that our approach does not rely at all on Number Theory. The price we pay is that our method produces N^ϵ losses. Results in [9] and [21] show that delicate use of Number Theory can remove the N^ϵ loss in some range of p .

2.3. The discrete restriction for lattice points on the sphere. Given integers $n \geq 3$ and $\lambda = N^2 \geq 1$ consider the discrete sphere

$$\mathcal{F}_{n,N^2} = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n : |\xi_1|^2 + \dots + |\xi_n|^2 = N^2\}.$$

In [7], the first author made the following conjecture about the eigenfunctions of the Laplacian on the torus and found some partial results

Conjecture 2.5. *For each $n \geq 3$, $a_\xi \in \mathbb{C}$, $\epsilon > 0$ and each $p \geq \frac{2n}{n-2}$ we have*

$$\left\| \sum_{\xi \in \mathcal{F}_{n,N^2}} a_\xi e(\xi \cdot x) \right\|_{L^p(\mathbb{T}^n)} \lesssim_\epsilon N^{\frac{n-2}{2} - \frac{n}{p} + \epsilon} \|a_\xi\|_{l^2(\mathcal{F}_{n,N^2})}. \quad (11)$$

We refer the reader to [11], [12] for a discussion on why the critical index $\frac{2n}{n-2}$ for the sphere is different from the one for the paraboloid. The conjecture has been verified by the authors in [11] for $p \geq \frac{2n}{n-3}$ when $n \geq 4$ and then later improved in [12] to $p \geq \frac{44}{7}$ when $n = 4$ and $p \geq \frac{14}{3}$ when $n = 5$. The methods in [7], [11] and [12] include Number Theory of various sorts, Incidence Geometry and Fourier Analysis. Using Theorem 2.2 we can further improve our results.

Theorem 2.6. *Let $n \geq 4$. The inequality (11) holds for $p \geq \frac{2(n-1)}{n-3}$.*

Proof Fix $\|a_\xi\|_2 = 1$ and define

$$F(x) = \sum_{\xi \in \mathcal{F}_{n,N^2}} a_\xi e(x \cdot \xi).$$

We start by recalling the following estimate (24) from [11], valid for $n \geq 4$ and $\alpha \gtrsim_\epsilon N^{\frac{n-1}{4} + \epsilon}$

$$|\{|F| > \alpha\}| \lesssim_\epsilon \alpha^{-2\frac{n-1}{n-3}} N^{\frac{2}{n-3}}. \quad (12)$$

By invoking interpolation with the trivial L^∞ bound, it suffices to consider the endpoint $p = p_n = \frac{2(n-1)}{n-3}$. Note that $\|F\|_\infty \leq N^{C_n}$. It follows that

$$\int |F|^{p_n} = \int_{N^{\frac{n-1}{4} + \epsilon} \lesssim_\epsilon \alpha \leq N^{C_n}} \alpha^{p_n-1} |\{|F| > \alpha\}| d\alpha + N^{(\frac{n-1}{4} + \epsilon)(p_n - \frac{2(n+1)}{n-1})} \int |F|^{\frac{2(n+1)}{n-1}}.$$

The result will follow by applying (12) to the first term and Theorem 2.2 with $p = \frac{2(n+1)}{n-1}$ to the second term. \blacksquare

2.4. Additive energies and Incidence Geometry. The proof of Theorem 1.1 in the following sections will implicitly rely on the incidence theory of tubes and cubes. This theory manifests itself in the deep multilinear Kakeya phenomenon which lies behind Theorem 6.1. It thus should come as no surprise that Theorem 1.1 has applications to Incidence Geometry.

An interesting question is whether there is a proof of Theorem 2.2 using softer arguments. Or at least if there is such an argument which recovers (6) for R large enough, depending on Λ . When $n = 3$ and $S = P^2$ we can prove such a result. In fact our result is surprisingly strong, in that the bound $|\Lambda|^\epsilon$ does not depend on the separation between the points in Λ .

Theorem 2.7. *Let $\Lambda \subset P^2$ be an arbitrary collection of distinct points. Then for R large enough, depending only on the geometry of Λ and on its cardinality $|\Lambda|$, we have*

$$\left(\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi \cdot x) \right|^4 \right)^{1/4} \lesssim_\epsilon |\Lambda|^\epsilon \|a_\xi\|_2. \quad (13)$$

Due to periodicity, this recovers (ii) of Theorem 2.3 for $n = 3$. To see the proof we recall some terminology and well known results.

Given an integer $k \geq 2$ and a set Λ in \mathbb{R}^n we introduce its k -energy

$$\mathbb{E}_k(\Lambda) = |\{(\lambda_1, \dots, \lambda_{2k}) \in \Lambda^{2k} : \lambda_1 + \dots + \lambda_k = \lambda_{k+1} + \dots + \lambda_{2k}\}|.$$

Note the trivial lower bound $|\mathbb{E}_k(\Lambda)| \geq |\Lambda|^k$.

We recall the point-line incidence theorem due to Szemerédi and Trotter

Theorem 2.8 ([27]). *There are $O(|\mathcal{L}| + |\mathcal{P}| + (|\mathcal{L}||\mathcal{P}|)^{2/3})$ incidences between any collections \mathcal{L} and \mathcal{P} of lines and points in the plane.*

Up to extra logarithmic factors, the same thing is conjectured to hold if lines are replaced with circles. Another related conjecture is

Conjecture 2.9 (The unit distance conjecture). *The number of unit distances between N points in the plane is always $\lesssim_\epsilon N^{1+\epsilon}$*

The point-circle and the unit distance conjectures are thought to be rather difficult, and only partial results are known.

Proof [of Theorem 2.7]

The following parameter encodes the "additive geometry" of Λ

$$v := \min \{ |\eta_1 + \eta_2 - \eta_3 - \eta_4| : \eta_i \in \Lambda \text{ and } |\eta_1 + \eta_2 - \eta_3 - \eta_4| \neq 0 \}.$$

We show that Theorem 2.7 holds if $R \gtrsim \frac{|\Lambda|^2}{v}$. Fix such an R .

Using restricted type interpolation it suffices to prove

$$\frac{1}{|B_R|} \int_{B_R} \left| \sum_{\eta \in \Lambda'} e(x \cdot \eta) \right|^4 dx \lesssim_\epsilon |\Lambda'|^{2+\epsilon},$$

for each subset $\Lambda' \subset \Lambda$. See Section 6 in [12] for details on this type of approach.

Expanding the L^4 norm we need to prove

$$\left| \sum_{\eta_i \in \Lambda'} \frac{1}{R^3} \int_{B_R} e((\eta_1 + \eta_2 - \eta_3 - \eta_4) \cdot x) dx \right| \lesssim_\epsilon |\Lambda'|^{2+\epsilon}.$$

Note that if $A \neq 0$

$$\left| \int_{-R}^R e(At) dt \right| \leq A^{-1}.$$

Using this we get that

$$\left| \sum_{\substack{\eta_i \in \Lambda' \\ |\eta_1 + \eta_2 - \eta_3 - \eta_4| \neq 0}} \frac{1}{R^3} \int_{B_R} e((\eta_1 + \eta_2 - \eta_3 - \eta_4) \cdot x) dx \right| \leq \frac{|\Lambda'|^4}{Rv} \leq |\Lambda'|^2.$$

Thus it suffices to prove the following estimate for the additive energy

$$\mathbb{E}_2(\Lambda') \lesssim_\epsilon |\Lambda'|^{2+\epsilon}. \quad (14)$$

Assume

$$\eta_1 + \eta_2 = \eta_3 + \eta_4, \quad (15)$$

with $\eta_i := (\alpha_i, \beta_i, \alpha_i^2 + \beta_i^2)$. It has been observed in [9] that given $A, B, C \in \mathbb{R}$, the equality

$$\eta_1 + \eta_2 = (A, B, C)$$

implies that for $i \in \{1, 2\}$

$$\left(\alpha_i - \frac{A}{2}\right)^2 + \left(\beta_i - \frac{B}{2}\right)^2 = \frac{2C - A^2 - B^2}{4}. \quad (16)$$

Thus the four points $P_i = (\alpha_i, \beta_i)$ corresponding to any additive quadruple (15) must belong to a circle. As observed in [9], this is enough to conclude (14) in the lattice case, as circles of radius M contain $\lesssim_\epsilon M^\epsilon$ lattice points. The bound (14) also follows immediately if one assumes the circle-point incidence conjecture.

We need however a new observation. Note that if (15) holds then in fact both P_1, P_2 and P_3, P_4 are diametrically opposite on the circle (16). Thus each additive quadruple gives rise to a distinct right angle, the one subtended by P_1, P_2, P_3 (say). The estimate (14) is then an immediate consequence of the following application of the Szemerédi-Trotter Theorem. ■

Theorem 2.10 (Pach, Sharir, [23]). *The number of repetitions of a given angle among N points in the plane is $O(N^2 \log N)$.*

It has been recognized that the restriction theory for the sphere and the paraboloid are very similar³. Consequently, one expects not only Theorem 2.7 to be true also for S^2 , but for a very similar argument to work in that case, too. If that is indeed the case, it does not appear to be obvious. The same argument as above shows that an additive quadruple of points on S^2 will belong to a circle on S^2 , and moreover the four points will be diametrically opposite in pairs. There will thus be at least $\mathbb{E}_2(\Lambda)$ right angles in Λ . This is however of no use in this setting, as Λ lives in three dimensions. It is proved in [1] that a set of N points in \mathbb{R}^3 has $O(N^{7/3})$ right angles, and moreover this bound is tight in general.

Another idea is to map an additive quadruple to the plane using the stereographic projection. The resulting four points will again belong to a circle, so the bound on the energy would follow if the circle-point incidence conjecture is proved. Unfortunately, the stereographic projection does not preserve the property of being diametrically opposite and thus prevents the application of Theorem 2.10. We thus ask

Question 2.11. *Is it true that $\mathbb{E}_2(\Lambda) \lesssim_\epsilon |\Lambda|^{2+\epsilon}$ for each finite $\Lambda \subset S^2$?*

One can ask the same question for P^{n-1} and S^{n-1} when $n \geq 4$. The right conjecture seems to be

$$\mathbb{E}_2(\Lambda) \lesssim_\epsilon |\Lambda|^{\frac{3n-5}{n-1}+\epsilon}. \quad (17)$$

³A notable difference is the lattice case of the discrete restriction, but that has to do with a rather specialized scenario

Interestingly, when $\Lambda \subset P^3$ this follows from the aforementioned result in [1], and in fact there is no Λ^ϵ loss this time. However, in the same paper [1] it is proved that this argument fails in dimensions five and higher: there is a set with N points in \mathbb{R}^4 which determines $\gtrsim N^3$ right angles. We point out that Theorem 2.2 implies (17) for subsets of P^{n-1} and S^{n-1} when $n \geq 3$, in the case when the points Λ are $\sim |\Lambda|^{-\frac{1}{n-1}}$ separated.

It is also natural to investigate the two dimensional phenomenon.

Question 2.12. *Is it true that for each $\Lambda \subset S$*

$$\mathbb{E}_3(\Lambda) \lesssim_\epsilon |\Lambda|^{3+\epsilon} \quad (18)$$

Surprisingly, this question seems to be harder than its three dimensional analogue from Theorem 2.7. Note that the case when the points are $|\Lambda|^{-C}$ separated follows from Theorem 2.2. We are not aware of an alternative (softer) argument.

A positive answer to Question 2.12 would have surprising applications to Number Theory. In particular it would answer the following question posed in [6].

Question 2.13. *Let N be a positive integer. Does (18) hold when Λ are the lattice points on the circle $N^{1/2}S^1$?*

Note that Theorem 2.2 is too weak to answer this question. Indeed, rescaling by $N^{1/2}$, the lattice points in $N^{1/2}S^1$ become $N^{-1/2}$ -separated points on S^1 . However, it is known that there are $O(N^{\frac{O(1)}{\log \log N}})$ lattice points on the circle $N^{1/2}S^1$.

The analysis in [6] establishes some partial results as well as some intriguing connections to the theory of elliptic curves, see for example Theorem 8 there. An easier question with similar flavor is answered in the next subsection.

The best that can be said regarding Question 2.12 with topological based methods seems to be the following

Proposition 2.14. *Let S be either P^1 or S^1 . For each $\Lambda \subset S$*

$$\mathbb{E}_3(\Lambda) \lesssim_\epsilon |\Lambda|^{\frac{7}{2}+\epsilon}.$$

Proof This was observed by Bombieri and the first author [6] when $S = S^1$. The proofs for P^1 and S^1 are very similar, we briefly sketch the details for $S = P^1$. Let N be the cardinality of Λ . It goes back to [9] that if

$$(x_1, x_1^2) + (x_2, x_2^2) + (x_3, x_3^2) = (n, j), \quad (19)$$

then the point $(3(x_1 + x_2), \sqrt{3}(x_1 - x_2))$ belongs to the circle centered at $(2n, 0)$ and of radius squared equal to $6j - 2n^2$. Note that there are N^2 such points with $(x_i, x_i^2) \in \Lambda$, call this set of points T . Assume we have M_n such circles containing roughly 2^n points $(3(x_1 + x_2), \sqrt{3}(x_1 - x_2)) \in T$ in such a way that (19) is satisfied for some $x_3 \in \Lambda$. Then clearly

$$\mathbb{E}_3(S) \lesssim \sum_{2^n \leq N} M_n 2^{2^n}.$$

It is easy to see that

$$M_n 2^n \lesssim N^3, \quad (20)$$

as each point in T can belong to at most N circles.

The nontrivial estimate is

$$M_n 2^{3n} \lesssim N^4, \quad (21)$$

which is an immediate consequence of the Szemerédi-Trotter Theorem for curves satisfying the following two fundamental axioms: two curves intersect in $O(1)$ points, and there are $O(1)$ curves passing through any two given points. The number of incidences between such curves and points is the same as in the case of lines and points, see for example Theorem 8.10 in [28]. Note that since our circles have centers on the x axis, any two points in T sitting in the upper (or lower) half plane determine a unique circle. Combining the two inequalities we get for each n

$$M_n 2^{2n} \lesssim N^{\frac{7}{2}}.$$

■

In the case when $\Lambda \subset S^1$, the same argument leads to incidences between unit circles and points. The outcome is the same, since for any two points there are at most two unit circles passing through them. An interesting observation is the fact that Question 2.12 has a positive answer if the Unit Distance Conjecture is assumed. Indeed, the argument above presents us with a collection T of N^2 points and a collection of $\lesssim N^3$ unit circles. For $2^n \lesssim N$ let M_n be the number of such circles with $\sim 2^n$ points. There will be at least $M_n 2^n$ unit distances among the N^2 points and the M_n centers. The Unit Distances Conjecture forces $M_n 2^n \lesssim_\epsilon (M_n + N^2)^{1+\epsilon}$. Since $M_n \lesssim N^3$, it immediately follows that $M_n 2^{2n} \lesssim_\epsilon N^{3+\epsilon}$ which gives the desired bound on the energy.

It seems likely that in order to achieve the conjectured bound on $\mathbb{E}_3(\Lambda)$, the structure of T must be exploited, paving the way to algebraic methods. One possibility is to make use of the fact that T has sumset structure. Another interesting angle for the parabola is the following. Recall that whenever (19) holds, the three points $(3(x_i + x_j), \sqrt{3}(x_i - x_j))$, $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$, belong to the circle centered at $(2n, 0)$ and of radius squared equal to $6j - 2n^2$. One can easily check that in fact they form an equilateral triangle! This potentially opens up the new toolbox of symmetries, since, for example, the rotation by $\pi/3$ about the center of any such circle C will preserve $C \cap T$.

2.5. Additive energies of annular sets. We start by mentioning a more general version of Theorem 2.2.

Theorem 2.15. *Let S be a C^2 compact hypersurface in \mathbb{R}^n with positive definite second fundamental form. For each $\theta \in \mathcal{P}_\delta$ let Λ_θ be a collection of points in θ and let $\Lambda = \cup_\theta \Lambda_\theta$. Then for each R -ball B_R with $R \gtrsim \delta^{-1}$ we have*

$$\left\| \sum_{\xi \in \Lambda} a_\xi e(x \cdot \xi) \right\|_{L^{\frac{2(n+1)}{n-1}}(B_R)} \lesssim_\epsilon \delta^{-\epsilon} \left(\sum_\theta \left\| \sum_{\xi \in \Lambda_\theta} a_\xi e(x \cdot \xi) \right\|_{L^{\frac{2(n+1)}{n-1}}(w_{B_R})}^2 \right)^{1/2}.$$

To see why this holds, note first that the case $R \sim \delta^{-1}$ follows by applying (the localized version of) Theorem 1.1 to functions whose Fourier transforms approximate weighted sums of Dirac deltas supported on Λ . The case $R \gtrsim \delta^{-1}$ then follows using Minkowski's inequality.

For $R > 1$ define

$$A_R = \{\xi \in \mathbb{R}^2 : R \leq |\xi| \leq R + R^{-1/3}\}$$

and $A'_R = A_R \cap \mathbb{Z}^2$. We prove the following inequality related to Question 2.13.

Theorem 2.16.

$$\mathbb{E}_3(A'_R) \lesssim_\epsilon |A'_R|^{3+\epsilon}. \quad (22)$$

Note that this is essentially sharp. The old Van der Corput estimate

$$|N(R) - \pi R^2| = o(R^{2/3})$$

for the error term in the Gauss circle problem shows that $|A'_R| = 2\pi R^{2/3} + o(R^{2/3})$. It thus suffices to show

$$\left\| \sum_{\xi \in A'_R} e(\xi \cdot x) \right\|_{L^6(\mathbb{T}^2)} \lesssim_\epsilon R^{\frac{1}{3}+\epsilon}.$$

Subdivide A_R into sectors A_α of size $\sim R^{1/3} \times R^{-1/3}$, so that each of them fits inside a rectangle R_α of area $< \frac{1}{2}$. Applying Theorem 2.15 after rescaling by R and using periodicity we get

$$\left\| \sum_{\xi \in A'_R} e(\xi \cdot x) \right\|_{L^6(\mathbb{T}^2)} \lesssim_\epsilon R^\epsilon \left(\sum_\alpha \left\| \sum_{\xi \in A'_\alpha} e(\xi \cdot x) \right\|_{L^6(\mathbb{T}^2)}^2 \right)^{1/2}, \quad (23)$$

with $A'_\alpha = A_\alpha \cap \mathbb{Z}^2$.

An elementary observation which goes back (at least) to Jarnick's work is that the area determined by a nondegenerate triangle with vertices in \mathbb{Z}^2 is half an integer. It follows that the points in each A'_α lie on a line L_α . In fact they must be equidistant, with consecutive points at distant d , for some $d \geq 1$. Define now for $1 \leq 2^s \lesssim R^{1/3}$

$$\mathcal{L}_s = \{\alpha : 2^s \leq |A'_\alpha| < 2^{s+1}\}.$$

Let also $\mathcal{L}_{s,m}$ be those $\alpha \in \mathcal{L}_s$ for which $2^m \leq d < 2^{m+1}$. Note that if $\alpha \in \mathcal{L}_{s,m}$ then L_α makes an angle $\sim 2^{-m}2^{-s}R^{-1/3}$ with the long axis of R_α . Thus the directions of the lines L_α will be distinct for each collection of $\alpha \in \mathcal{L}_{s,m}$ whose corresponding arcs on S^1 are $C2^{-m}2^{-s}R^{-1/3}$ -separated. Obviously there are $u, v \in A'_\alpha$ such that $|u - v| \sim 2^m$. Since there are $O(2^{2m})$ lattice points with length $\sim 2^m$, it follows that there can be at most $O(2^{2m})$ elements α in $\mathcal{L}_{s,m}$ which are $C2^{-m}2^{-s}R^{-1/3}$ -separated. Thus $|\mathcal{L}_{s,m}| \lesssim 2^{2m}R^{1/3}2^{-m}2^{-s}$. As $2^{m+s} \lesssim R^{1/3}$ we conclude that $|\mathcal{L}_s| \lesssim R^{2/3}2^{-2s}$.

Note that by using Hölder's inequality with $L^2 - L^\infty$ endpoints we have

$$\left\| \sum_{\xi \in A'_\alpha} e(\xi \cdot x) \right\|_{L^6(\mathbb{T}^2)} \leq |A'_\alpha|^{5/6}.$$

Using this, the bound on $|\mathcal{L}_s|$ and (23) finishes the proof of (22).

2.6. Counting solutions of Diophantine inequalities. In this section we show how to use the Decoupling Theorem to recover and generalize results from the literature as well as to prove some new type of results. We do not aim at providing a systematic study of these problems but rather to explain the way our methods become useful in this context.

To motivate our first application we consider the system of equations for $k \geq 2$

$$\begin{cases} n_1^k + n_2^k + n_3^k &= n_4^k + n_5^k + n_6^k \\ n_1 + n_2 + n_3 &= n_4 + n_5 + n_6 \end{cases},$$

with $1 \leq n_i \leq N$. It is easy to see that there are $6N^3$ trivial solutions. The question here is to determine the correct asymptotic for the number $U_k(N)$ of nontrivial solutions. This is in part motivated by connections to the Waring problem, see [3]. The case $k = 3$ known as the Segre cubic has been intensely studied. Vaughan and Wooley have proved in [30] that $U_3(N) \sim N^2(\log N)^5$, see also [14] for a more precise result. For $k \geq 4$, Greaves [20] (see also [26]) has proved that $U_k(N) = O(N^{\frac{17}{6}+\epsilon})$. All these results follow through the use of rather delicate Number Theory.

While our methods in this paper can not produce such fine estimates, they successfully address the perturbed case. The following result is perhaps a surprising consequence of Theorem 2.2.

Theorem 2.17. *For fixed $k \geq 2$ and C the system*

$$\begin{cases} |n_1^k + n_2^k + n_3^k - n_4^k - n_5^k - n_6^k| \leq CN^{k-2} \\ n_1 + n_2 + n_3 = n_4 + n_5 + n_6 \end{cases}$$

has $O(N^{3+\epsilon})$ solutions with $n_i \sim N$.

Proof Apply Theorem 2.2 to the curve

$$\{(\xi, \xi^k) : |\xi| \sim 1\},$$

the points

$$\Lambda = \{(\frac{n}{N}, (\frac{n}{N})^k) : n \sim N\}$$

and $\delta = N^{-2}$. We get that

$$\frac{1}{N^4} \int_{|x| \leq N^2} \int_{|y| \leq N^2} \left| \sum_{n \sim N} e(x \frac{n}{N} + y (\frac{n}{N})^k) \right|^6 dx dy \lesssim_\epsilon N^{3+\epsilon}.$$

Upon rescaling and using periodicity we get

$$\begin{aligned} N^{k-3} \int_{|x| \leq N} \int_{|y| \leq N^{2-k}} \left| \sum_{n \sim N} e(xn + yn^k) \right|^6 dx dy = \\ N^{k-2} \int_{|x| \leq 1} \int_{|y| \leq N^{2-k}} \left| \sum_{n \sim N} e(xn + yn^k) \right|^6 dx dy \lesssim_\epsilon N^{3+\epsilon}. \end{aligned} \quad (24)$$

Let now $\phi : \mathbb{R} \rightarrow [0, \infty)$ be a Schwartz function with positive Fourier transform satisfying $\widehat{\phi}(\xi) \gtrsim 1$ for $|\xi| \leq 1$. Define $\phi_N(y) = \phi(N^{k-2}y)$. A standard argument allows us to replace the cutoff $|y| \leq N^{2-k}$ with $\phi_N(y)$ in (24). It suffices then to note that

$$\begin{aligned} N^{k-2} \int_{|x| \leq 1} \int_{\mathbb{R}} \left| \sum_{n \sim N} e(xn + yn^k) \right|^6 \phi_N(y) dx dy = \\ \sum_{\substack{n_i \sim N \\ n_1 + n_2 + n_3 = n_4 + n_5 + n_6}} \widehat{\phi}(N^{2-k}(n_1^k + n_2^k + n_3^k - n_4^k - n_5^k - n_6^k)). \end{aligned}$$

■

Note also that our method proves that

$$N^{k-2} \int_{|x| \leq 1} \int_{|y-c| \leq N^{2-k}} \left| \sum_{n \sim N} e(xn + yn^k) \right|^6 dx dy \lesssim_\epsilon N^{3+\epsilon},$$

for each $c \in \mathbb{R}$. The difficulty in proving this for $k \geq 3$ using purely number theoretic methods comes from estimating the contribution of the minor arcs. When $k = 2$ the left hand side is at least $cN^3 \log N$, which shows that one can not dispense with the N^ϵ term. This can be seen by evaluating the contribution from the major arcs, see for example page 118 in [9].

Our second application generalizes the result from [25] ($k = 4$) to $k \geq 4$. Its original motivation lies in the study of the Riemann zeta function on the critical line (cf. [4], [5]) and also in getting refinements of Heath-Brown's variant of Weyl's inequality, see [25].

Theorem 2.18. *For $k \geq 4$ and $0 \leq \lambda \leq 1$ we have*

$$\int_{|x| \leq 1} \int_0^\lambda \left| \sum_{n \sim N} e(xn^2 + yn^k) \right|^6 dx dy \lesssim_\epsilon \lambda N^{3+\epsilon} + N^{4-k+\epsilon}.$$

In particular, the system

$$\begin{cases} |n_1^k + n_2^k + n_3^k - n_4^k - n_5^k - n_6^k| \leq CN^{k-1} \\ n_1^2 + n_2^2 + n_3^2 = n_4^2 + n_5^2 + n_6^2 \end{cases}$$

has $O(N^{3+\epsilon})$ solutions with $n_i \sim N$.

Proof The estimate on the number of solutions follows by using $\lambda = N^{1-k}$. Note that it suffices to prove that

$$\int_{|x| \leq 1} \int_J \left| \sum_{n \sim N} e(xn^2 + yn^k) \right|^6 dx dy \lesssim_\epsilon N^{4-k+\epsilon},$$

for each interval J with length N^{1-k} .

We apply Theorem 2.15 to the curve

$$\{(\xi^2, \xi^k) : |\xi| \sim 1\},$$

the points

$$\Lambda = \left\{ \left(\left(\frac{n}{N} \right)^2, \left(\frac{n}{N} \right)^k \right) : n \sim N \right\},$$

$R^{-1} = \delta = N^{-1}$ and $B_N = [MN, (M+1)N] \times N^k J$ with $M \in \{-N, \dots, 0, \dots, N-1\}$. Summing over M we get due to periodicity

$$\begin{aligned} & \left\| \sum_{n \sim N} e\left(x' \frac{n^2}{N^2} + y' \frac{n^k}{N^k}\right) \right\|_{L^6(|x'| \leq N^2, y' \in N^k J)} \lesssim_\epsilon \\ & N^\epsilon \left(\sum_\alpha \left\| \sum_{n \in I_\alpha} e\left(x' \frac{n^2}{N^2} + y' \frac{n^k}{N^k}\right) \right\|_{L^6(|x'| \leq N^2, y' \in N^k J)}^2 \right)^{1/2}. \end{aligned}$$

Here $I_\alpha = [n_\alpha, n_\alpha + N^{1/2}]$ are intervals of length $N^{1/2}$ that partition the integers $n \sim N$. It follows after a change of variables that

$$\begin{aligned} & \left\| \sum_{n \sim N} e(xn^2 + yn^k) \right\|_{L^6(|x| \leq 1, y \in J)} \lesssim_\epsilon \\ & N^\epsilon \left(\sum_\alpha \left\| \sum_{n \in I_\alpha} e(xn^2 + yn^k) \right\|_{L^6(|x| \leq 1, y \in J)}^2 \right)^{1/2}. \end{aligned} \tag{25}$$

Next note that for $y \in J$

$$\begin{aligned} & \left| \sum_{n \in I_\alpha} e(xn^2 + yn^k) \right| = \\ & \left| \sum_{m=1}^{N^{1/2}} c_{m,J,n_\alpha} e\left(m^2\left(x + \frac{k(k-1)}{2}n_\alpha^{k-2}y\right) + m(2xn_\alpha + kn_\alpha^{k-1}y)\right) \right| + O(1), \end{aligned}$$

with $|c_{m,J,n_\alpha}| = 1$. To estimate the first term we change variables to

$$\begin{cases} x' = x + \frac{k(k-1)}{2}n_\alpha^{k-2}y \\ y' = (2k - k^2)n_\alpha^{k-1}y \end{cases}.$$

We get

$$\begin{aligned} & \left\| \sum_{n \in I_\alpha} e(xn^2 + yn^k) \right\|_{L^6(|x| \leq 1, y \in J)} \lesssim \\ & n_\alpha^{-\frac{k-1}{6}} \left\| \sum_{m=1}^{N^{1/2}} c_{m,J,n_\alpha} e(x'm^2 + 2x'n_\alpha m + my') \right\|_{L^6(B_C)} + O(N^{\frac{1-k}{6}}) = \\ & n_\alpha^{-\frac{k-1}{6}} \left\| \sum_{m=1+n_\alpha}^{N^{1/2}+n_\alpha} c_{m,J,n_\alpha} e(x'm^2 + my') \right\|_{L^6(B_C)} + O(N^{\frac{1-k}{6}}), \end{aligned}$$

for some ball B_C of radius $C = O(1)$. This can further be seen to be $O(N^{\frac{1}{4} + \frac{1-k}{6} + \epsilon})$ by the result in Theorem 2.3. We conclude that (25) is $O(N^{\frac{1}{2} + \frac{1-k}{6} + \epsilon})$, as desired.

There are further number theoretical consequences of the decoupling theory that will be investigated elsewhere. \blacksquare

3. NORMS AND WAVE PACKET DECOMPOSITIONS

We will use C to denote various constants that are allowed to depend on n, p, α , but never on δ . $|\cdot|$ will denote both the Lebesgue measure on \mathbb{R}^n and the cardinality of finite sets.

This section and the next one is concerned with introducing some of the tools that will be used in the proof of Theorem 1.1 from Section 6. For $2 \leq p \leq \infty$ we define the norm

$$\|f\|_{p,\delta} = \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^2 \right)^{1/2},$$

where f_θ is the Fourier restriction of f to θ . We note the following immediate consequence of Hölder's inequality

$$\|f\|_{p,\delta} \leq \|f\|_{2,\delta}^{\frac{2}{p}} \|f\|_{\infty,\delta}^{1-\frac{2}{p}} \quad (26)$$

and the fact that if $\text{supp}(\hat{f}) \subset \mathcal{N}_\delta$ then

$$\|f\|_{2,\delta} \sim \|f\|_2.$$

Definition 3.1. Let N be a real number greater than 1. An N -tube T is an $N^{1/2} \times \dots \times N^{1/2} \times N$ rectangular parallelepiped in \mathbb{R}^n which has dual orientation to some $\theta = \theta(T) \in \mathcal{P}_\delta$. We call a collection of N -tubes separated if no more than C tubes with a given orientation overlap.

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by

$$\phi(x) = (1 + |x|^2)^{-M},$$

for some M large enough compared to n , whose value will become clear from the argument. Define $\phi_T = \phi \circ a_T$, where a_T is the affine function mapping T to the unit cube in \mathbb{R}^n centered at the origin.

Definition 3.2. An N -function is a function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$f = \sum_{T \in \mathcal{T}(f)} f_T$$

where $\mathcal{T}(f)$ consists of finitely many separated N -tubes T and moreover

$$|f_T| \leq \phi_T,$$

$$\|f_T\|_p \sim |T|^{1/p}, \quad 1 \leq p \leq \infty$$

and

$$\text{supp}(\widehat{f_T}) \subset \theta(T).$$

For $\theta \in \mathcal{P}_{1/N}$ let $\mathcal{T}(f, \theta)$ denote the N -tubes in $\mathcal{T}(f)$ dual to θ . An N -function is called balanced if $|\mathcal{T}(f, \theta)| \leq 2|\mathcal{T}(f, \theta')|$ whenever $\mathcal{T}(f, \theta), \mathcal{T}(f, \theta') \neq \emptyset$.

The $\|\cdot\|_{p,\delta}$ norms of N -functions are asymptotically determined by their plate distribution over the sectors θ .

Lemma 3.3. For each N -function f and for $2 \leq p \leq \infty$

$$\|f\|_{p,1/N} \sim N^{\frac{n+1}{2p}} \left(\sum_{\theta} |\mathcal{T}(f, \theta)|^{\frac{2}{p}} \right)^{1/2}. \quad (27)$$

If the N -function is balanced then

$$\|f\|_{p,1/N} \sim N^{\frac{n+1}{2p}} M(f)^{\frac{1}{2} - \frac{1}{p}} |\mathcal{T}(f)|^{1/p}, \quad (28)$$

where $M(f)$ is the number of sectors θ for which $\mathcal{T}(f, \theta) \neq \emptyset$.

Proof It suffices to prove (27) when $\mathcal{T}(f) = \mathcal{T}(f, \theta)$ for some θ . We first observe the trivial estimates $\|f\|_1 \lesssim |T| |\mathcal{T}(f)|$, $\|f\|_\infty \lesssim 1$ and $\|f\|_2 \sim |T|^{1/2} |\mathcal{T}(f)|^{1/2}$. Applying Hölder's inequality twice we get

$$\|f\|_2^{\frac{2(p-1)}{p}} \|f\|_1^{\frac{2-p}{p}} \leq \|f\|_p \leq \|f\|_1^{1/p} \|f\|_\infty^{1/p'},$$

which is exactly what we want. ■

The crucial role played by balanced N -functions is encoded by

Lemma 3.4. (i) Each N -function f can be written as the sum of $O(\log |\mathcal{T}(f)|)$ balanced N -functions.

(ii) For each balanced N -function f and $2 \leq p \leq \infty$ we have the converse of (26), namely

$$\|f\|_{p,1/N} \sim \|f\|_{2,1/N}^{\frac{2}{p}} \|f\|_{\infty,1/N}^{1-\frac{2}{p}}. \quad (29)$$

Proof Note that (i) is immediate by using dyadic ranges. Also, (ii) will follow from (28). \blacksquare

In the remaining sections we will use the fact that the contribution of f to various inequalities comes from logarithmically many N -functions. The basic mechanism is the following.

Lemma 3.5 (Wave packet decomposition). Assume f is Fourier supported in \mathcal{N}_δ . Then for each dyadic $0 < \lambda \lesssim \|f\|_{\infty,\delta}$ there is an $N = \delta^{-1}$ -function f_λ such that

$$f = \sum_{\lambda \lesssim \|f\|_{\infty,\delta}} \lambda f_\lambda$$

and for each $2 \leq p < \infty$ we have

$$\lambda^p N^{\frac{n+1}{2}} |\mathcal{T}(f_\lambda)| \leq \|\lambda f_\lambda\|_{p,\delta}^p \lesssim \|f\|_{p,\delta}^p. \quad (30)$$

Proof Using a partition of unity write

$$f = \sum_{\theta \in \mathcal{P}_\delta} \tilde{f}_\theta$$

with $\tilde{f}_\theta = f_\theta * K_\theta$ Fourier supported in $\frac{9}{10}\theta$ with $\|K_\theta\|_1 \lesssim 1$. Consider a windowed Fourier series expansion for each \tilde{f}_θ

$$\tilde{f}_\theta = \sum_{T \in \mathcal{T}_\theta} \langle \tilde{f}_\theta, \varphi_T \rangle \varphi_T,$$

where φ_T are L^2 normalized Schwartz functions Fourier localized in θ such that

$$|T|^{1/2} |\varphi_T| \lesssim \phi_T.$$

The tubes in \mathcal{T}_θ are separated. Note that by Hölder's inequality

$$|a_T := \frac{1}{|T|^{1/2}} \langle \tilde{f}_\theta, \varphi_T \rangle| \lesssim \|\tilde{f}_\theta\|_\infty \lesssim \|f_\theta\|_\infty \leq \|f\|_{\infty,\delta}.$$

It is now clear that we should take

$$f_\lambda = \sum_{\theta} \sum_{T \in \mathcal{T}_\theta: |a_T| \sim \lambda} a_T \lambda^{-1} |T|^{1/2} \varphi_T.$$

To see (30) note that the first inequality follows from (27) and the fact that $\|\cdot\|_{l^{p/2}} \leq \|\cdot\|_{l^1}$. To derive the second inequality, it suffices to prove that for each θ

$$\left\| \sum_{T \in \mathcal{T}_\theta: |a_T| \sim \lambda} \langle \tilde{f}_\theta, \varphi_T \rangle \varphi_T \right\|_p \lesssim \|f_\theta\|_p.$$

Using (27) and the immediate consequence of Hölder's inequality $|a_T|^p \lesssim \int |\tilde{f}_\theta|^p |T|^{-1/2} \varphi_T$ we get

$$\begin{aligned} & \left\| \sum_{T \in \mathcal{T}_\theta: |a_T| \sim \lambda} \langle \tilde{f}_\theta, \varphi_T \rangle \varphi_T \right\|_p^p \lesssim \lambda^p |T| |\{T \in \mathcal{T}_\theta : |a_T| \sim \lambda\}| \lesssim \\ & \lesssim |T| \sum_{T \in \mathcal{T}_\theta} |a_T|^p \lesssim \int |\tilde{f}_\theta|^p \sum_{T \in \mathcal{T}_\theta} \phi_T \lesssim \int |\tilde{f}_\theta|^p \lesssim \int |f_\theta|^p. \end{aligned}$$

■

4. PARABOLIC RESCALING

Proposition 4.1. *Let $\delta \leq \sigma < \frac{1}{2}$ and $K_p(\frac{\delta}{\sigma})$ be such that*

$$\|f\|_p \leq K_p\left(\frac{\delta}{\sigma}\right) \left(\sum_{\theta \in \mathcal{P}_{\frac{\delta}{\sigma}}} \|f_\theta\|_p^2 \right)^{1/2},$$

for each f with Fourier support in $\mathcal{N}_{\frac{\delta}{\sigma}}$. Then for each f with Fourier support in \mathcal{N}_δ and for each $\tau \in \mathcal{P}_\sigma$ we have

$$\|f_\tau\|_p \lesssim K_p\left(\frac{\delta}{\sigma}\right) \left(\sum_{\theta \in \mathcal{P}_\delta: \theta \cap \tau \neq \emptyset} \|f_\theta\|_p^2 \right)^{1/2}.$$

Proof Let $a = (a_1, \dots, a_{n-1})$ be the center of the $\sigma^{1/2}$ -cube C_τ , see (1). We will perform the parabolic rescaling via the linear transformation

$$L_\tau(\xi_1, \dots, \xi_n) = (\xi'_1, \dots, \xi'_n) = \left(\frac{\xi_1 - a_1}{\sigma^{1/2}}, \dots, \frac{\xi_{n-1} - a_{n-1}}{\sigma^{1/2}}, \frac{\xi_n - 2 \sum_{i=1}^{n-1} a_i \xi_i + \sum_{i=1}^{n-1} a_i^2}{\sigma} \right).$$

Note that

$$\xi'_n - \sum_{i=1}^{n-1} \xi_i'^2 = \sigma^{-1} \left(\xi_n - \sum_{i=1}^{n-1} \xi_i^2 \right).$$

It follows that L_τ maps the Fourier support $\mathcal{N}_\delta \cap \tau$ of f_τ to $\mathcal{N}_{\frac{\delta}{\sigma}} \cap ([-\frac{1}{2}, \frac{1}{2}]^{n-1} \times \mathbb{R})$. Also, for each $\tau' \in \mathcal{P}_{\frac{\delta}{\sigma}}$ we have that $L_\tau(\theta) = \tau'$ for some $\theta \in \mathcal{P}_\delta$ with $\theta \cap \tau \neq \emptyset$. Thus

$$\|f_\tau\|_p^p = \|g\|_p^p (\det(L_\tau))^{1-p},$$

where g is the L_τ dilation of f_τ Fourier supported in $\mathcal{N}_{\frac{\delta}{\sigma}} \cap ([-\frac{1}{2}, \frac{1}{2}]^{n-1} \times \mathbb{R})$. By invoking the hypothesis we get that

$$\|g\|_p \lesssim K_p\left(\frac{\delta}{\sigma}\right) \left(\sum_{\tau' \in \mathcal{P}_{\frac{\delta}{\sigma}}} \|g_{\tau'}\|_p^2 \right)^{1/2}.$$

We are done if we use the fact that

$$\|f_\theta\|_p^p = \|g_{\tau'}\|_p^p (\det(L_\tau))^{1-p}$$

whenever $L_\tau(\theta) = \tau'$. ■

5. LINEAR VERSUS MULTILINEAR DECOUPLING

Let $g : P^{n-1} \rightarrow \mathbb{C}$. For a cap τ on P^{n-1} we let $g_\tau = g|_\tau$ be the (spatial) restriction of g to τ . We denote by $\pi : P^{n-1} \rightarrow [-1/2, 1/2]^{n-1}$ the projection map.

Definition 5.1. *We say that the caps τ_1, \dots, τ_n on P^{n-1} are ν -transverse if the volume of the parallelepiped spanned by any unit normals v_i at τ_i is greater than ν .*

We denote by $C_{p,n}(\delta, \nu)$ the smallest constant such that

$$\left\| \left(\prod_{i=1}^n |\widehat{g_{\tau_i} d\sigma}| \right)^{1/n} \right\|_{L^p(B_{\delta^{-1}})} \leq C_{p,n}(\delta, \nu) \left[\prod_{i=1}^n \left(\sum_{\substack{\theta: \delta^{1/2}\text{-cap} \\ \theta \subset \tau_i}} \|\widehat{g_\theta d\sigma}\|_{L^p(w_{B_{\delta^{-1}}})}^2 \right)^{1/2} \right]^{1/n},$$

for each ν -transverse caps $\tau_i \subset P^{n-1}$, each δ^{-1} ball $B_{\delta^{-1}}$ and each $g : P^{n-1} \rightarrow \mathbb{C}$.

Let also $K_{p,n}(\delta)$ be the smallest constant such that

$$\|\widehat{gd\sigma}\|_{L^p(B_{\delta^{-1}})} \leq K_{p,n}(\delta) \left(\sum_{\theta: \delta^{1/2}\text{-cap}} \|\widehat{g_\theta d\sigma}\|_{L^p(w_{B_{\delta^{-1}}})}^2 \right)^{1/2},$$

for each $g : P^{n-1} \rightarrow \mathbb{C}$ and each δ^{-1} ball $B_{\delta^{-1}}$.

Remark 5.2. As before, the norm $\|f\|_{L^p(w_{B_R})}$ refers to the weighted L^p integral

$$\left(\int_{\mathbb{R}^n} |f(x)|^p w_{B_R}(x) dx \right)^{1/p}$$

for some weight satisfying (8). It is important to realize that there are such weights which in addition are Fourier supported in $B(0, R^{-1})$. Note also that if g is supported on P^{n-1} and if $\widehat{w_{B_R}}$ is supported in $B(0, R^{-1})$, then $(\widehat{gd\sigma})w_{B_R}$ has Fourier support inside $\mathcal{N}_{R^{-1}}$. This simple observation justifies the various (entirely routine) localization arguments that follow, as well as the interplay between Fourier transforms of functions and Fourier transforms of measures supported on P^{n-1} . In particular let $K_{p,n}^{(1)}(\delta)$ be the smallest constant such that

$$\|\widehat{gd\sigma}\|_{L^p(w_{B_{\delta^{-1}}})} \leq K_{p,n}^{(1)}(\delta) \left(\sum_{\theta: \delta^{1/2}\text{-cap}} \|\widehat{g_\theta d\sigma}\|_{L^p(w_{B_{\delta^{-1}}})}^2 \right)^{1/2},$$

for each $g : P^{n-1} \rightarrow \mathbb{C}$ and each δ^{-1} ball $B_{\delta^{-1}}$. Then $K_{p,n}^{(1)}(\delta) \sim_{n,p} K_{p,n}(\delta)$. Also, if $K_{p,n}^{(2)}(\delta)$, $K_{p,n}^{(3)}(\delta)$, $K_{p,n}^{(4)}(\delta)$ are the smallest constants such that

$$\|f\|_{L^p(\mathbb{R}^n)} \leq K_{p,n}^{(2)}(\delta) \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2},$$

$$\|f\|_{L^p(B_{\delta^{-1}})} \leq K_{p,n}^{(3)}(\delta) \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2},$$

$$\|f\|_{L^p(w_{B_{\delta^{-1}}})} \leq K_{p,n}^{(4)}(\delta) \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_{L^p(w_{B_{\delta^{-1}}})}^2 \right)^{1/2},$$

for each f Fourier supported in \mathcal{N}_δ and each δ^{-1} ball $B_{\delta^{-1}}$, then

$$K_{p,n}^{(2)}(\delta), K_{p,n}^{(3)}(\delta), K_{p,n}^{(4)}(\delta) \sim_{n,p} K_{p,n}(\delta).$$

The same observation applies to the family of constants related to $C_{p,n}(\delta, \nu)$ from the multilinear inequality.

Note that due to Hölder's inequality

$$C_{p,n}(\delta, \nu) \leq K_{p,n}(\delta).$$

We will show that the reverse inequality essentially holds true.

Theorem 5.3. *Assume one of the following holds*

(i) $n = 2$

(ii) $n \geq 3$ and $K_{p,d}(\delta') \lesssim_{\epsilon} \delta'^{-\epsilon}$ for each $\delta', \epsilon > 0$ and each $2 \leq d \leq n - 1$.

Then for each $0 < \nu \leq 1$ there is $\epsilon(\nu)$ with $\lim_{\nu \rightarrow 0} \epsilon(\nu) = 0$ and C_{ν} such that

$$K_{p,n}(\delta) \leq C_{\nu} \delta^{-\epsilon(\nu)} C_{p,n}(\delta, \nu)$$

for each δ .

We prove the case $n = 3$ and will indicate the modifications needed for $n \geq 4$. The argument will also show how to deal with the case $n = 2$.

Remark 5.4. If $Q_1, Q_2, Q_3 \in [-1/2, 1/2]^2$ the volume of parallelepiped spanned by the unit normals to P^2 at $\pi^{-1}(Q_i)$ is comparable to the area of the triangle $\Delta Q_1 Q_2 Q_3$.

The key step in the proof of Theorem 5.3 for $n = 3$ is the following.

Proposition 5.5. *Assume $K_{p,2}(\delta) \lesssim_{\epsilon} \delta^{-\epsilon}$ for each $\epsilon > 0$. Then for each ϵ there is C_{ϵ} such that for each $R > 1$ and $K \geq 1$*

$$\begin{aligned} \|\widehat{gd\sigma}\|_{L^p(w_{B_R})} &\leq C_{\epsilon} K^{\epsilon} \left[\left(\sum_{\substack{\alpha \subset P^2 \\ \alpha: \frac{1}{K} - \text{cap}}} \|\widehat{g_{\alpha} d\sigma}\|_{L^p(w_{B_R})}^2 \right)^{1/2} + \left(\sum_{\substack{\beta \subset P^2 \\ \beta: \frac{1}{K^{1/2}} - \text{cap}}} \|\widehat{g_{\beta} d\sigma}\|_{L^p(w_{B_R})}^2 \right)^{1/2} \right] + \\ &\quad + K^{10} C_{p,3}(R^{-1}, K^{-2}) \left(\sum_{\substack{\Delta \subset P^2 \\ \Delta: \frac{1}{R^{1/2}} - \text{cap}}} \|\widehat{g_{\Delta} d\sigma}\|_{L^p(w_{B_R})}^2 \right)^{1/2} \end{aligned}$$

Proof Following the formalism in [13] we will regard $|\widehat{g_{\alpha} d\sigma}|$ as being essentially constant on each ball B_K . Denote by $c_{\alpha}(B_K)$ this value and let α^* be the cap that maximizes it.

The starting point in the argument is the observation in [13] that for each B_K there exists a line $L = L(B_K)$ in the (ξ_1, ξ_2) plane such that if

$$S_L = \{(\xi_1, \xi_2) : \text{dist}((\xi_1, \xi_2), L) \leq \frac{C}{K}\}$$

then for $x \in B_K$

$$\begin{aligned} |\widehat{gd\sigma}(x)| &\leq \\ C \max_{\alpha} |\widehat{g_{\alpha} d\sigma}(x)| &+ \end{aligned} \tag{31}$$

$$K^4 \max_{\substack{\alpha_1, \alpha_2, \alpha_3 \\ K^{-2} - \text{transverse}}} \left(\prod_{i=1}^3 |\widehat{g_{\alpha_i} d\sigma}(x)| \right)^{1/3} + \tag{32}$$

$$| \sum_{\alpha \subset \pi^{-1}(S_L) \cap P^2} \widehat{g_\alpha d\sigma}(x) |. \quad (33)$$

To see this, we distinguish three scenarios. First, if $c_\alpha(B_K) \leq K^{-2}c_{\alpha^*}(B_K)$ for each α with $\text{dist}(\pi(\alpha), \pi(\alpha^*)) \geq \frac{10}{K}$, then (31) suffices, as

$$|\widehat{gd\sigma}(x)| \leq \sum_{\alpha} c_\alpha(B_K).$$

Otherwise, there is α^{**} with $\text{dist}(\pi(\alpha^{**}), \pi(\alpha^*)) \geq \frac{10}{K}$ and $c_{\alpha^{**}}(B_K) \geq K^{-2}c_{\alpha^*}(B_K)$. The line L is determined by α^*, α^{**} .

Second, if there is α^{***} such that $\pi(\alpha^{***})$ intersects the complement of S_L and $c_{\alpha^{***}}(B_K) \geq K^{-2}c_{\alpha^*}(B_K)$ then (32) suffices. Indeed, note that $\alpha^*, \alpha^{**}, \alpha^{***}$ are K^{-2} transverse by Remark 5.4.

Otherwise the sum of (31) and (33) will suffice.

The only nontrivial case to address is the one corresponding to this latter scenario. Cover $\pi^{-1}(S_L) \cap P^2$ by pairwise disjoint strips U of length $\sim \frac{1}{K^{1/2}}$. An application of Fubini's Theorem shows that

$$\| \sum_{\alpha: \pi(\alpha) \subset S_L} \widehat{g_\alpha d\sigma} \|_{L^p(B_K)} \lesssim K_{p,2}(K^{-1}) (\sum_U \| \widehat{g_U d\sigma} \|_{L^p(w_{B_K})}^2)^{1/2}.$$

We are of course relying on the fact that $\pi^{-1}(L)$ is a parabola with principal curvature equal to 1. Note however that since we are dealing with the third scenario

$$(\sum_U \| \widehat{g_U d\sigma} \|_{L^p(B_K)}^2)^{1/2} \lesssim (\sum_{\substack{\beta: \frac{1}{K^{1/2}} - \text{cap}: \\ \pi(\beta) \subset S_L}} \| \widehat{g_\beta d\sigma} \|_{L^p(w_{B_K})}^2)^{1/2} + \| \widehat{g_{\alpha^*} d\sigma} \|_{L^p(w_{B_K})}.$$

We conclude that in either case

$$\begin{aligned} \| \widehat{gd\sigma} \|_{L^p(B_K)} &\leq C_\epsilon K^\epsilon [(\sum_{\substack{\alpha \subset P^2 \\ \alpha: \frac{1}{K} \text{ cap}}} \| \widehat{g_\alpha d\sigma} \|_{L^p(w_{B_K})}^2)^{1/2} + (\sum_{\substack{\beta \subset P^2 \\ \beta: \frac{1}{K^{1/2}} \text{ cap}}} \| \widehat{g_\beta d\sigma} \|_{L^p(w_{B_K})}^2)^{1/2}] + \\ &\quad + K^{10} C_{p,3}(R^{-1}, K^{-2}) (\sum_{\substack{\Delta \subset P^2 \\ \Delta: \frac{1}{R^{1/2}} \text{ cap}}} \| \widehat{g_\Delta d\sigma} \|_{L^p(w_{B_K})}^2)^{1/2} \end{aligned}$$

It suffices now to raise to the p^{th} power and sum over $B_K \subset B_R$ using Minkowski's inequality. Also, the norm $\| \widehat{gd\sigma} \|_{L^p(B_R)}$ can be replaced by the weighted norm $\| \widehat{gd\sigma} \|_{L^p(w_{B_R})}$ via the localization argument described in Remark 5.2. \blacksquare

Rescaling gives the following.

Proposition 5.6. *Let τ be a δ cap. Assume $K_{p,2}(\delta') \lesssim_\epsilon \delta'^{-\epsilon}$ for each $\epsilon > 0$ and δ' . Then for each ϵ there is C_ϵ such that for each $R > \delta^{-2}$ and $K \geq 1$*

$$\begin{aligned} \| \widehat{g_\tau d\sigma} \|_{L^p(w_{B_R})} &\leq C_\epsilon K^\epsilon [(\sum_{\substack{\alpha \subset \tau \\ \alpha: \frac{\delta}{K} \text{ cap}}} \| \widehat{g_\alpha d\sigma} \|_{L^p(w_{B_R})}^2)^{1/2} + (\sum_{\substack{\beta \subset \tau \\ \beta: \frac{\delta}{K^{1/2}} \text{ cap}}} \| \widehat{g_\beta d\sigma} \|_{L^p(w_{B_R})}^2)^{1/2}] + \\ &\quad K^{10} C_{p,3}((R\delta^2)^{-1}, K^{-2}) (\sum_{\substack{\Delta \subset \tau \\ \Delta: \frac{1}{R^{1/2}} \text{ cap}}} \| \widehat{g_\Delta d\sigma} \|_{L^p(w_{B_R})}^2)^{1/2} \end{aligned}$$

Proof Note that if $\gamma \in [-1/2, 1/2]^2$ then

$$\widehat{g_{\pi^{-1}\gamma} d\sigma}(x_1, x_2, x_3) = \int_{B_\eta(c)} \pi g(\xi_1, \xi_2) e(\xi_1 x_1 + \xi_2 x_2 + (\xi_1^2 + \xi_2^2) x_3) d\xi_1 d\xi_2.$$

Let $a = (a_1, a_2)$. Changing variable to $\xi_i = a_i + \delta \xi'_i$ and letting

$$\begin{aligned} \pi g^{a,\delta}(\xi') &= \pi g(a + \delta \xi') \\ \gamma' &= \delta^{-1}(\gamma - a) \end{aligned}$$

we get

$$|\widehat{g_{\pi^{-1}\gamma} d\sigma}(x_1, x_2, x_3)| = \delta^2 |\widehat{g_{\pi^{-1}\gamma'}^{a,\delta} d\sigma}(\delta(x_1 + 2a_1 x_3), \delta(x_2 + 2a_2 x_3), \delta^2 x_3)|.$$

In particular

$$\|\widehat{g_{\pi^{-1}\gamma} d\sigma}\|_{L^p(w_{B_R})} = \delta^{2-\frac{4}{p}} \|\widehat{g_{\pi^{-1}\gamma'}^{a,\delta} d\sigma}\|_{L^p(w_{C_R})},$$

where C_R is a $\sim \delta R \times \delta R \times \delta^2 R$ cylinder. Cover C_R with balls $B_{\delta^2 R}$. The result now follows by applying Proposition 5.5 to $g^{a,\delta}$ (with a the center of $\pi(\tau)$) on each $B_{\delta^2 R}$ and then summing using Minkowski's inequality. \blacksquare

We are now ready to prove Theorem 5.3 for $n = 3$. Let $K = \nu^{-1/2}$. Iterate Proposition 5.6 starting with scale $\delta = 1$ until we reach scale $\delta = R^{-1/2}$. Each iteration lowers the scale of the caps from δ to at least $\frac{\delta}{K^{1/2}}$. Thus we have to iterate $\sim \log_K R$ times. Since

$$C_{p,3}((\delta^2 R)^{-1}, K^{-2}) \lesssim C_{p,3}(R^{-1}, \nu)$$

we get for each $\epsilon > 0$

$$\begin{aligned} \|\widehat{g d\sigma}\|_{L^p(w_{B_R})} &\leq (CC_\epsilon K^\epsilon)^{\log_K R} K^{10} C_{p,3}(R^{-1}, \nu) \left(\sum_{\Delta: \frac{1}{R^{1/2}} \text{ cap}} \|\widehat{g_\Delta d\sigma}\|_{L^p(w_{B_R})}^2 \right)^{1/2} = \\ &R^{-2 \log_\nu (CC_\epsilon) + \epsilon} \nu^{-5} C_{p,3}(R^{-1}, \nu) \left(\sum_{\Delta: \frac{1}{R^{1/2}} \text{ cap}} \|\widehat{g_\Delta d\sigma}\|_{L^p(w_{B_R})}^2 \right)^{1/2}. \end{aligned}$$

The result follows since C, C_ϵ do not depend on ν .

To summarize, the proof of Theorem 5.3 for $n = 3$ relied on the hypothesis that the contribution coming from caps living near the intersection of P^2 with a plane is controlled by $K_{p,2}(\delta) = O(\delta^{-\epsilon})$. When $n \geq 4$, the hypothesis $K_{p,d}(\delta) = O(\delta^{-\epsilon})$ for $2 \leq d \leq n-1$ plays the same role, it controls the contribution coming from caps living near lower dimensional elliptic paraboloids with principal curvatures equal to 1. And of course, no such hypothesis is needed when $n = 2$. The statement and the proof of Proposition 5.6 for these values of n will hold without further modifications.

6. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1 for P^{n-1} . We first consider the open range $p > \frac{2(n+1)}{n-1}$, and in the end of the section we prove the result for the endpoint. We use notation from the previous section such as $K_{p,n}(\delta)$, $C_{p,n}(\delta, \nu)$ and $\delta = N^{-1}$.

Fix $p > \frac{2(n+1)}{n-1}$. Proposition 4.1 shows that $K_{p,n}(\delta) \lesssim K_{p,n}(\delta^{1/2})^2$. Let

$$\gamma = \liminf_{\delta \rightarrow 0} \frac{\log K_{p,n}(\delta)}{\log(\delta^{-1})}.$$

It follows that for each ϵ

$$\delta^{-\gamma} \lesssim K_{p,n}(\delta) \lesssim_{\epsilon} \delta^{-\gamma-\epsilon}.$$

Write $\gamma = \frac{n-1}{4} - \frac{n+1}{2p} + \alpha$. For the rest of the argument we will assume that $\alpha > 0$, and will show how to reach a contradiction.

The following multilinear restriction estimate from [2] will play a key role in our proof.

Theorem 6.1. *Let τ_1, \dots, τ_n be ν -transverse caps on P^{n-1} and assume $\widehat{f_i}$ is supported on the δ -neighborhood of τ_i . Then we have*

$$\|(\prod_{i=1}^n |f_i|)^{1/n}\|_{L^{\frac{2n}{n-1}}(B_N)} \lesssim_{\epsilon, \nu} N^{-\frac{1}{2}+\epsilon} (\prod_{i=1}^n \|f_i\|_{L^2})^{1/n}.$$

As observed in [10], using Plancherel's identity this easily implies that

$$C_{p,n}(\delta, \nu) \lesssim_{\epsilon} \delta^{-\epsilon}$$

for $p = \frac{2n}{n-1}$.

Define

$$\xi = \frac{2}{(p-2)(n-1)}$$

$$\eta = \frac{n(np-2n-p-2)}{2p(n-1)^2(p-2)}.$$

Since $p > \frac{2(n+1)}{n-1}$ we have that $\xi < \frac{1}{2}$. A simple computation reveals that the assumption $\alpha > 0$ is equivalent with

$$\gamma \frac{1-\xi}{1-2\xi} > \frac{n-1}{4} - \frac{n^2+n}{2p(n-1)} + \frac{2\eta}{1-2\xi}.$$

It follows that we can choose $s_0 \in \mathbb{N}$ large enough and ϵ_0 small enough so that

$$\gamma \left(\frac{1-\xi}{1-2\xi} - \frac{\xi(2\xi)^{s_0}}{1-2\xi} \right) >$$

$$\frac{n-1}{4} - \frac{n^2+n}{2p(n-1)} + 2^{s_0}\epsilon_0 + \frac{2\eta}{1-2\xi}(1-(2\xi)^{s_0}) + \frac{n}{(n-1)p}(2\xi)^{s_0}. \quad (34)$$

Choose $\nu > 0$ small enough such that $\epsilon_0 > \epsilon(\nu)$, with $\epsilon(\nu)$ as in Theorem 5.3. Note that s_0 , ϵ_0 and ν depend only on the fixed parameters p, n, α . As a result, we follow our convention and do not record the dependence on them when using the symbol \lesssim .

Throughout the rest of the section ν , s_0 and ϵ_0 will always refer to these values. To simplify notation we let $K(\delta) := K_{p,n}(\delta)$ and $C_{p,n}(\delta, \nu) = C(\delta)$.

We now present the first piece of our argument.

Proposition 6.2. *Let τ_1, \dots, τ_n be ν -transverse caps on P^{n-1} and assume \widehat{f}_i is supported on the δ -neighborhood of τ_i . Then we have*

$$\|(\prod_{i=1}^n |f_i|)^{1/n}\|_{L^p(B_N)} \lesssim_\epsilon N^{\frac{n-1}{4} - \frac{n^2+n}{2p(n-1)} + \epsilon} (\prod_{i=1}^n \|f_i\|_{\frac{p(n-1)}{n}, \delta})^{\frac{1}{n}}. \quad (35)$$

Proof Assume first that $\|f_i\|_{\infty, \delta} = 1$ for each i . Let $\lambda_n = N^{\frac{n-1}{4}}$ and $F = (\prod_{i=1}^n |f_i|)^{1/n}$. Note that

$$\|F\|_\infty \leq \lambda_n.$$

Then using Theorem 6.1 and Hölder's inequality we have

$$\|F\|_{L^p(B_N)}^p \leq \lambda_n^{p - \frac{2n}{n-1}} \|F\|_{L^{\frac{2n}{n-1}}(B_N)}^{\frac{2n}{n-1}} \lesssim_\epsilon \lambda_n^{p - \frac{2n}{n-1}} N^{\epsilon - \frac{n}{n-1}} (\prod_{i=1}^n \|f_i\|_{L^2})^{\frac{2}{n-1}}.$$

By invoking homogeneity we find that

$$\|F\|_{L^p(B_N)} \lesssim_\epsilon \lambda_n^{1 - \frac{2n}{(n-1)p}} N^{\epsilon - \frac{n}{(n-1)p}} (\prod_{i=1}^n (\|f_i\|_2^{\frac{2n}{(n-1)p}} \|f_i\|_{\infty, \delta}^{1 - \frac{2n}{(n-1)p}}))^{\frac{1}{n}} \quad (36)$$

holds true without the restriction $\|f_i\|_{\infty, \delta} = 1$.

Finally, to get (35), we use the wave packet decomposition and the fact that

$$\|f\|_2^{\frac{2n}{(n-1)p}} \|f\|_{\infty, \delta}^{1 - \frac{2n}{(n-1)p}} \sim \|f\|_{\frac{p(n-1)}{n}, \delta}$$

if f is a balanced N -function. We can assume $\|f_i\|_{\frac{p(n-1)}{n}, \delta} = 1$ for each i . Write like in Lemma 3.5

$$f_i = \sum_{\lambda \lesssim \|f_i\|_{\infty, \delta}} \lambda f_{i, \lambda}.$$

We use the triangle inequality to estimate the left hand side of (35). In the following C will denote a large enough constant depending on n, p . As $\|f_{i, \lambda}\|_\infty \lesssim 1$, we have that

$$\|f_{i, \lambda}\|_{L^p(B_N)} \leq N^C.$$

As the right hand side in (35) is $\gtrsim N^{-C}$, it follows that the contribution coming from $\lambda f_{i, \lambda}$ with $\lambda \lesssim N^{-C}$ is well controlled.

On the other hand, recall that by Bernstein's inequality, $\|f_i\|_{\infty, \delta} \lesssim N^C$. This shows that it suffices to consider $O(\log \delta^{-1})$ many terms in the triangle inequality. Each of these terms is dealt with by using (36), Lemma 3.4 and (30). \blacksquare

At this point it is useful to introduce the local norms for $g : P^{n-1} \rightarrow \mathbb{C}$ and arbitrary balls B

$$\|\widehat{gd\sigma}\|_{p, \delta, B} = (\sum_{\theta \text{ } \delta^{1/2}\text{-cap}} \|\widehat{g_\theta d\sigma}\|_{L^p(w_B)}^2)^{1/2}.$$

Remark 5.2 and (35) show that

$$\|(\prod_{i=1}^n |\widehat{g_i d\sigma}|)^{1/n}\|_{L^p(B_N)} \lesssim_\epsilon N^{\frac{n-1}{4} - \frac{n^2+n}{2p(n-1)} + \epsilon} (\prod_{i=1}^n \|\widehat{g_i d\sigma}\|_{\frac{p(n-1)}{n}, \delta, B_N})^{\frac{1}{n}}, \quad (37)$$

for each $g_i : \tau_i \rightarrow \mathbb{C}$, where τ_i are as before.

Next we iterate (37) and invoke Theorem 6.1 at each step of the iteration. In the following, we describe the first step of the iteration scheme.

First, Hölder's inequality implies that

$$\|\widehat{g_i d\sigma}\|_{\frac{p(n-1)}{n}, \delta, B_N} \leq \|\widehat{g_i d\sigma}\|_{p, \delta, B_N}^{1 - \frac{2}{(p-2)(n-1)}} \|\widehat{g_i d\sigma}\|_{2, \delta, B_N}^{\frac{2}{(p-2)(n-1)}}.$$

In particular we have

$$\|(\prod_{i=1}^n |\widehat{g_i d\sigma}|)^{1/n}\|_{L^p(B_N)} \lesssim_\epsilon N^{\frac{n-1}{4} - \frac{n^2+n}{2p(n-1)} + \epsilon} \left(\prod_{i=1}^n (\|\widehat{g_i d\sigma}\|_{p, \delta, B_N}^{1 - \frac{2}{(p-2)(n-1)}} \|\widehat{g_i d\sigma}\|_{2, \delta, B_N}^{\frac{2}{(p-2)(n-1)}}) \right)^{\frac{1}{n}}. \quad (38)$$

Consider a finitely overlapping cover of B_N by balls Δ with radius $N^{1/2}$. Note that

$$\|(\prod_{i=1}^n |\widehat{g_i d\sigma}|)^{1/n}\|_{L^p(B_N)} \lesssim \left(\sum_{\Delta} \|(\prod_{i=1}^n |\widehat{g_i d\sigma}|)^{1/n}\|_{L^p(\Delta)}^p \right)^{1/p}.$$

We will use (38) at scale $\delta^{1/2}$ to bound each $\|(\prod_{i=1}^n |\widehat{g_i d\sigma}|)^{1/n}\|_{L^p(\Delta)}$. After raising to the p^{th} power, the right hand side of (38) is summed using Hölder's inequality

$$\sum_{\Delta} b_{\Delta}^{\theta p} \prod_{i=1}^n a_{\Delta, i}^{\frac{1-\theta}{n} p} \leq \left(\sum_{\Delta} b_{\Delta}^p \right)^{\theta} \prod_{i=1}^n \left(\sum_{\Delta} a_{\Delta, i}^p \right)^{\frac{1-\theta}{n}}.$$

To sum the factors $a_{\Delta, i}^p = \|\widehat{g_i d\sigma}\|_{p, \delta^{1/2}, \Delta}^p$ that appear both in the first and second term of (38) we invoke first Minkowski's inequality then Proposition 4.1 and get

$$\sum_{\Delta} \|\widehat{g_i d\sigma}\|_{p, \delta^{1/2}, \Delta}^p \lesssim \|\widehat{g_i d\sigma}\|_{p, \delta^{1/2}, B_N}^p \lesssim K(\delta^{1/2})^p \|\widehat{g_i d\sigma}\|_{p, \delta, B_N}^p.$$

We next show how to sum the factors $b_{\Delta}^p = (\prod_{i=1}^n \|\widehat{g_i d\sigma}\|_{2, \delta^{1/2}, \Delta}^p)^{1/n}$. Rather than using the n -linear Hölder's and then Minkowski's inequality as we did with the terms $a_{\Delta, i}^n$, we transform b_{Δ}^p to make it amenable to another application of Theorem 6.1.

To this end we recall the standard formalism (see e.g. [13]) that for each $\delta^{1/2}$ -cap θ , $|\widehat{g_{\theta} d\sigma}|$ is essentially constant on each Δ . Thus, in particular it is easy to see that

$$\begin{aligned} \sum_{\Delta \subset B_N} \prod_{i=1}^n \|\widehat{g_i d\sigma}\|_{p, \delta, \Delta}^p &\lesssim \sum_{\Delta \subset B_N} \left\| \prod_{i=1}^n \left(\sum_{\substack{\theta \text{ } \delta^{1/2}\text{-cap} \\ \theta \subset \tau_i}} |\widehat{g_{i, \theta} d\sigma}|^2 \right)^{\frac{1}{2n}} \right\|_{L^p(w_{\Delta})}^p \lesssim \\ &\lesssim \left\| \prod_{i=1}^n \left(\sum_{\substack{\theta \text{ } \delta^{1/2}\text{-cap} \\ \theta \subset \tau_i}} |\widehat{g_{i, \theta} d\sigma}|^2 \right)^{\frac{1}{2n}} \right\|_{L^p(w_{B_N})}^p. \end{aligned} \quad (39)$$

Next, a randomization argument and Theorem 6.1 imply that

$$\left\| \left[\prod_{i=1}^n \left(\sum_{\theta \in \mathcal{P}_{\delta}} |f_{i, \theta}|^2 \right)^{1/2} \right]^{1/n} \right\|_{L^{\frac{2n}{n-1}}(B_N)} \lesssim_\epsilon N^{-\frac{1}{2} + \epsilon} \left(\prod_{i=1}^n \|f_i\|_2 \right)^{1/n},$$

whenever \widehat{f}_i is supported in a δ -neighborhood of τ_i . Combining this with the trivial inequality

$$\|[\prod_{i=1}^n (\sum_{\theta \in \mathcal{P}_\delta} |f_{i,\theta}|^2)^{1/2}]^{1/n}\|_{L^\infty(B_N)} \leq (\prod_{i=1}^n \|f_i\|_{\infty,\delta})^{1/n}$$

then with Hölder's inequality gives

$$\|[\prod_{i=1}^n (\sum_{\theta \in \mathcal{P}_\delta} |f_{i,\theta}|^2)^{1/2}]^{1/n}\|_{L^p(B_N)} \lesssim_\epsilon N^{-\frac{n}{(n-1)p} + \epsilon} (\prod_{i=1}^n (\|f\|_2^{\frac{2n}{(n-1)p}} \|f_i\|_{\infty,\delta}^{1 - \frac{2n}{(n-1)p}}))^{1/n}.$$

Reasoning like in the proof of Proposition 6.2, we get that

$$\|[\prod_{i=1}^n (\sum_{\theta \in \mathcal{P}_\delta} |f_{i,\theta}|^2)^{1/2}]^{1/n}\|_{L^p(B_N)} \lesssim_\epsilon N^{-\frac{n}{(n-1)p} + \epsilon} (\prod_{i=1}^n \|f_i\|_{\frac{(n-1)p}{n}, \delta})^{1/n}, \quad (40)$$

for each \widehat{f}_i is supported in a δ -neighborhood of τ_i . Now (41), (39) and (40) lead to

$$\sum_{\Delta \subset B_N} \prod_{i=1}^n \|\widehat{g_i d\sigma}\|_{2,\delta,\Delta}^{\frac{p}{n}} \lesssim_\epsilon N^{-\frac{n}{(n-1)} + \epsilon} N^{\frac{n}{2}(\frac{p}{2}-1)} (\prod_{i=1}^n \|\widehat{g_i d\sigma}\|_{\frac{(n-1)p}{n}, \delta, B_N})^{1/n}.$$

Putting all these together, we have actually proved that

$$\begin{aligned} & \|(\prod_{i=1}^n \widehat{g_i d\sigma})^{1/n}\|_{L^p(B_N)} \lesssim_\epsilon \\ & K(\delta^{1/2})^{1 - \frac{2}{(p-2)(n-1)}} N^{\frac{n-1}{8} - \frac{n^2+n}{4p(n-1)} + [-\frac{n}{(n-1)} + \frac{n}{2}(\frac{p}{2}-1)] \frac{2}{p(p-2)(n-1)} + \epsilon} \times \\ & \times (\prod_{i=1}^n (\|\widehat{g_i d\sigma}\|_{p,\delta,B_N}^{1 - \frac{2}{(p-2)(n-1)}} \|\widehat{g_i d\sigma}\|_{\frac{(n-1)p}{n}, \delta, B_N}^{\frac{2}{(p-2)(n-1)}}))^{1/n}. \end{aligned}$$

This inequality represents the first step of our iteration. It is important to make the following observation, that may be obscured by the complexity of various exponents of N . The ultimate gain in our argument comes from the way we handle the term $\|\widehat{g_i d\sigma}\|_{2,\delta^{\frac{1}{2}},\Delta}$ appearing in (38). Note that by orthogonality followed by Hölder's inequality, for each Δ

$$\|\widehat{g_i d\sigma}\|_{2,\delta^{\frac{1}{2}},\Delta} \lesssim \|\widehat{g_i d\sigma}\|_{2,\delta,\Delta} \leq N^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|\widehat{g_i d\sigma}\|_{p,\delta,\Delta}. \quad (41)$$

If we had relied instead on just (41), this would have led to the proof of Theorem 1.1 in some partial range.

For a fixed θ consider the inequality

$$\|(\prod_{i=1}^n |\widehat{g_i d\sigma}|)^{1/n}\|_{L^p(B_N)} \lesssim_{\epsilon,\theta} A_\theta(N) N^\epsilon X(B_N)^{1-\theta} Y(B_N)^\theta, \quad (42)$$

for arbitrary $\epsilon > 0$, N , g_i and B_N as before. Here we simplify notation and write

$$X(B_N) = (\prod_{i=1}^n \|\widehat{g_i d\sigma}\|_{p,\delta,B_N})^{\frac{1}{n}},$$

$$Y(B_N) = \left(\prod_{i=1}^n \|g_i d\sigma\|_{\frac{p(n-1)}{n}, \delta, B_N} \right)^{\frac{1}{n}}.$$

What we proved so far can be summarized as follows.

Proposition 6.3. (a) Inequality (42) holds true for $\theta = 1$ with $A_1(N) = N^{\frac{n-1}{4} - \frac{n^2+n}{2p(n-1)}}$.
(b) Moreover, if we assume (42) for some $\theta \in (0, 1]$, then we also have (42) for $\frac{2\theta}{(p-2)(n-1)}$ with

$$A_{\frac{2\theta}{(p-2)(n-1)}}(N) = A_\theta(N^{1/2}) \delta^{-\frac{\gamma}{2}(1 - \frac{2\theta}{(p-2)(n-1)})} N^{\frac{n(np-2n-p-2)\theta}{2p(n-1)^2(p-2)}}.$$

Proposition 6.3 implies that for each $s \geq 0$

$$A_{\xi^s}(N) = N^{\psi(\xi^s)}$$

with

$$\psi(\xi^{s+1}) = \frac{1}{2}\psi(\xi^s) + \frac{\gamma}{2}(1 - \xi^{s+1}) + \eta\xi^s. \quad (43)$$

Recall that $\xi < \frac{1}{2}$. Iterating (43) gives

$$\psi(\xi^s) = \frac{1}{2^s}\psi(1) + \gamma(1 - 2^{-s}) + 2\left(\frac{\eta}{\xi} - \frac{\gamma}{2}\right)\frac{2^{-s} - \xi^s}{\xi^{-1} - 2} \quad (44)$$

Note that $Y(B_N) \lesssim X(B_N)N^{\frac{n}{(n-1)p}}$. As (42) holds for $\theta = \xi^s$ and arbitrary ν -transverse caps τ_i we get

$$C(\delta) \lesssim_{\epsilon, s} \delta^{-\epsilon} A_{\xi^s}(N) N^{\frac{n\xi^s}{(n-1)p}}. \quad (45)$$

To finish the argument, we will argue using induction on n that $\alpha = 0$. We first consider $n = 2$. Since (45) (with $s = s_0$) holds for arbitrarily small δ and ϵ , using Theorem 5.3 we get

$$\gamma - \epsilon_0 \leq \psi(\xi^{s_0}) + \frac{n\xi^{s_0}}{(n-1)p}. \quad (46)$$

Combining (44) and (46) we find

$$\gamma\left(\frac{1-\xi}{1-2\xi} - \frac{\xi(2\xi)^{s_0}}{1-2\xi}\right) \leq \psi(1) + 2^{s_0}\epsilon_0 + \frac{2\eta}{1-2\xi}(1 - (2\xi)^{s_0}) + \frac{n}{(n-1)p}(2\xi)^{s_0},$$

which contradicts (34). Thus $\alpha = 0$ and Theorem 1.1 is proved for $n = 2$ and $p > 6$.

Assume now that $n \geq 3$ and that Theorem 1.1 was proved for all $2 \leq d \leq n-1$ when $p > \frac{2(d+1)}{d-1}$. To prove Theorem 1.1 in \mathbb{R}^n for $p > \frac{2(n+1)}{n-1}$, it suffices to prove it for $\frac{2(n+1)}{n-1} < p < \frac{2n}{n-2}$. Note that in this range we have $p < \frac{2(d+1)}{d-1}$ for each $2 \leq d \leq n-1$ and thus Theorem 5.3 is applicable, due to the induction hypothesis.

It remains to see why Theorem 1.1 holds for the endpoint $p = p_n = \frac{2(n+1)}{n-1}$. Remark 5.2 shows that it suffices to investigate the best constant in the localized inequality

$$\|f\|_{L^p(B_N)} \leq K_{p,n}^{(3)}(\delta) \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}, \quad (47)$$

for each N -ball B_N . It suffices now to invoke Theorem 1.1 for $p > \frac{2(n+1)}{n-1}$ together with

$$\|f\|_{L^{pn}(B_N)} \lesssim \|f\|_{L^p(B_N)} N^{\frac{n}{pn} - \frac{n}{p}} \quad (\text{by Hölder's inequality})$$

$$\|f_\theta\|_{L^p(\mathbb{R}^n)} \lesssim N^{\frac{n+1}{2p} - \frac{n+1}{2pn}} \|f_\theta\|_{L^{pn}(\mathbb{R}^n)} \quad (\text{by Bernstein's inequality}),$$

and then to let $p \rightarrow p_n$.

7. EXTENSION TO OTHER HYPERSURFACES

Let S be a compact C^2 hypersurface in \mathbb{R}^n with positive definite second fundamental form. Recall that we have proved Theorem 1.1 for P^{n-1} . By a linear transformation, the proof extends to elliptic paraboloids of the form

$$\{(\xi_1, \dots, \xi_{n-1}, \theta_1 \xi_1^2 + \dots + \theta_{n-1} \xi_{n-1}^2) \in \mathbb{R}^n : |\xi_i| \leq 1/2\},$$

with $\theta_i \in [C^{-1}, C]$. The implicit bounds will of course depend on $C > 0$.

We now show how to extend the result in Theorem 1.1 to S as above. It suffices to prove the result for $p = \frac{2(n+1)}{n-1}$. We can assume that all the principal curvatures of S are in $[C^{-1}, C]$.

The following argument is sketched in [19] and was worked out in detail for conical surfaces in [24]. For $\delta < 1$, let as before $K_p(\delta)$ be the smallest constant such that for each f with Fourier support in \mathcal{N}_δ we have

$$\|f\|_p \leq K_p(\delta) \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^2 \right)^{1/2}.$$

Fix such an f . First, note that

$$\|f\|_p \leq K_p(\delta^{\frac{2}{3}}) \left(\sum_{\tau \in \mathcal{P}_{\delta^{\frac{2}{3}}}} \|f_\tau\|_p^2 \right)^{1/2}.$$

Second, our assumption on the principal curvatures of S combined with Taylor's formula shows that on each $\tau \in \mathcal{P}_{\delta^{\frac{2}{3}}}$, S is within δ from a paraboloid with similar principal curvatures. By invoking Theorem 1.1 for this paraboloid, combined with parabolic rescaling (Proposition 4.1) we get

$$\|f_\tau\|_p \lesssim_\epsilon \delta^{-\epsilon} \left(\sum_{\theta \in \mathcal{P}_\delta : \theta \subset \tau} \|f_\theta\|_p^2 \right)^{1/2}.$$

For each $\epsilon > 0$, we conclude the existence of C_ϵ such that for each $\delta < 1$

$$K_p(\delta) \leq C_\epsilon \delta^{-\epsilon} K_p(\delta^{\frac{2}{3}}).$$

By iteration this immediately leads to $K_p(\delta) \lesssim_\epsilon \delta^{-\epsilon}$.

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